

DEFORMATIONS OF QUASICOHERENT SHEAVES OF ALGEBRAS

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ABSTRACT. Gerstenhaber and Schack ([GS]) developed a deformation theory of presheaves of algebras on small categories. We translate their cohomological description to sheaf cohomology. More precisely, we describe the deformation space of (admissible) quasicoherent sheaves of algebras on a quasiprojective scheme X in terms of sheaf cohomology on X and $X \times X$. These results are applied to the study of deformations of the sheaf D_X of differential operators on X . In particular, in case X is a flag variety we show that any deformation of D_X , which is induced by a deformation of \mathcal{O}_X , must be trivial. This result is used in [LR3], where we study the localization construction for quantum groups.

1. INTRODUCTION

Let X be a topological space, k be a field, and \mathcal{A}_X be a sheaf of k -algebras on X . We would like to study infinitesimal deformations of \mathcal{A}_X . Such deformations form a k -vector space which we denote by $\text{def}(\mathcal{A}_X)$. In case $X = pt$ it is well known that the infinitesimal deformations of (the k -algebra) $A = \mathcal{A}_X$ are controlled by the Hochschild cohomology of A . More precisely, $\text{def}(A) = HH^2(A) = \text{Ext}_{A \otimes A^o}^2(A, A)$. However, for a general X and \mathcal{A}_X the situation is more subtle. More generally, given an \mathcal{A}_X -bimodule \mathcal{M}_X we may ask for cohomological interpretation of $\text{exal}(\mathcal{A}_X, \mathcal{M}_X)$ – the space of algebra extensions of \mathcal{A}_X by \mathcal{M}_X ($\text{exal}(\mathcal{A}_X, \mathcal{A}_X) = \text{def}(\mathcal{A}_X)$).

Gerstenhaber and Schack ([GS]) developed a deformation theory of *presheaves* of algebras. Given a small category \mathcal{U} and a presheaf of algebras $\mathcal{A}_{\mathcal{U}}$ on \mathcal{U} (i.e. a contravariant functor from \mathcal{U} to the category of k -algebras) they consider the space $\text{def}(\mathcal{A}_{\mathcal{U}})$ of infinitesimal deformations of $\mathcal{A}_{\mathcal{U}}$ and give it a cohomological interpretation. Namely, given an $\mathcal{A}_{\mathcal{U}}$ -bimodule $\mathcal{M}_{\mathcal{U}}$ they define a natural exact sequence of complexes of k -vector spaces

$$0 \rightarrow T_a^{\bullet}(\mathcal{M}_{\mathcal{U}}) \rightarrow T^{\bullet}(\mathcal{M}_{\mathcal{U}}) \rightarrow \bar{T}^{\bullet}(\mathcal{M}_{\mathcal{U}}) \rightarrow 0.$$

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The middle term is the total complex of the *simplicial bar resolution* of $\mathcal{M}_{\mathcal{U}}$ and

$$H^i(T^\bullet(\mathcal{M}_{\mathcal{U}})) = \text{Ext}_{\mathcal{A}_{\mathcal{U}} \otimes \mathcal{A}_{\mathcal{U}}^o}^i(\mathcal{A}_{\mathcal{U}}, \mathcal{M}_{\mathcal{U}})$$

– the Hochschild cohomology of $\mathcal{A}_{\mathcal{U}}$ with coefficients in $\mathcal{M}_{\mathcal{U}}$. The cohomology $H^i(\bar{T}^\bullet(\mathcal{M}_{\mathcal{U}}))$ is the cohomology $H^i(\mathcal{U}, \mathcal{M}_{\mathcal{U}})$ of the nerve of \mathcal{U} (or the classifying space of \mathcal{U}) with coefficients in $\mathcal{M}_{\mathcal{U}}$. Finally,

$$H^2(T_a^\bullet(\mathcal{M}_{\mathcal{U}})) = \text{exal}(\mathcal{A}_{\mathcal{U}}, \mathcal{M}_{\mathcal{U}});$$

in particular, $H^2(T_a^\bullet(\mathcal{A}_{\mathcal{U}})) = \text{def}(\mathcal{A}_{\mathcal{U}})$. As a consequence they obtain a long exact sequence of k -spaces

$$\begin{array}{ccccccc} \dots & \rightarrow & \text{Ext}_{\mathcal{A}_{\mathcal{U}} \otimes \mathcal{A}_{\mathcal{U}}^o}^1(\mathcal{A}_{\mathcal{U}}, \mathcal{M}_{\mathcal{U}}) & \rightarrow & H^1(\mathcal{U}, \mathcal{M}_{\mathcal{U}}) & \rightarrow & \\ \text{exal}(\mathcal{A}_{\mathcal{U}}, \mathcal{M}_{\mathcal{U}}) & \rightarrow & \text{Ext}_{\mathcal{A}_{\mathcal{U}} \otimes \mathcal{A}_{\mathcal{U}}^o}^2(\mathcal{A}_{\mathcal{U}}, \mathcal{M}_{\mathcal{U}}) & \rightarrow & H^2(\mathcal{U}, \mathcal{M}_{\mathcal{U}}) & \rightarrow & \\ \dots & & & & & & \end{array}$$

Returning to our problem of trying to interpret cohomologically the space $\text{exal}(\mathcal{A}_X, \mathcal{M}_X)$ we may proceed as follows. Let \mathcal{U} be the category of (all or some) open subsets of X . From the sheaf of algebras \mathcal{A}_X and its bimodule \mathcal{M}_X we obtain the corresponding presheaves $\mathcal{A}_{\mathcal{U}}$ and $\mathcal{M}_{\mathcal{U}}$. At this point there are two natural questions.

Q1. Is $\text{exal}(\mathcal{A}_X, \mathcal{M}_X)$ equal to $\text{exal}(\mathcal{A}_{\mathcal{U}}, \mathcal{M}_{\mathcal{U}})$?

Q2. Can we interpret the spaces $\text{Ext}_{\mathcal{A}_{\mathcal{U}} \otimes \mathcal{A}_{\mathcal{U}}^o}^i(\mathcal{A}_{\mathcal{U}}, \mathcal{M}_{\mathcal{U}})$ and $H^i(\mathcal{U}, \mathcal{M}_{\mathcal{U}})$ as sheaf cohomologies on X or $X \times X$?

The answers to these questions in general are probably negative.

In this paper we obtain positive answers to the above questions in case X is a quasiprojective scheme over k and \mathcal{A}_X and \mathcal{M}_X are quasi-coherent sheaves on X , which satisfy some additional conditions (the pair $(\mathcal{A}_X, \mathcal{M}_X)$ must be admissible in the sense of Definition 4.7 below). In this case there is a natural quasicoherent sheaf of algebras \mathcal{A}_Y^e on the product scheme $Y = X \times X$ (this is the analogue of the ring $A \otimes A^o$ for a single algebra A). Moreover, the \mathcal{A}_X -bimodule \mathcal{M}_X gives rise to a \mathcal{A}_Y^e -module $\tilde{\mathcal{M}}_Y$; in particular, the \mathcal{A}_X -bimodule \mathcal{A}_X defines an \mathcal{A}_Y^e -module $\tilde{\mathcal{A}}_Y$. If \mathcal{U} is the category of all *affine* open subsets of X , then we prove that

$$\text{exal}(\mathcal{A}_X, \mathcal{M}_X) = \text{exal}(\mathcal{A}_{\mathcal{U}}, \mathcal{M}_{\mathcal{U}}),$$

and

$$\text{Ext}_{\mathcal{A}_{\mathcal{U}} \otimes \mathcal{A}_{\mathcal{U}}^o}^i(\mathcal{A}_{\mathcal{U}}, \mathcal{M}_{\mathcal{U}}) = \text{Ext}_{\mathcal{A}_Y^e}^i(\tilde{\mathcal{A}}_Y, \tilde{\mathcal{M}}_Y),$$

$$H^i(\mathcal{U}, \mathcal{M}_{\mathcal{U}}) = H^i(X, \mathcal{M}_X).$$

In particular, we obtain the long exact sequence

$$\begin{array}{ccccccc} \dots & \rightarrow & \mathrm{Ext}_{\mathcal{A}_Y^e}^1(\tilde{\mathcal{A}}_Y, \tilde{\mathcal{M}}_Y) & \rightarrow & H^1(X, \mathcal{M}_X) & \rightarrow & \\ \mathrm{exal}(\mathcal{A}_X, \mathcal{M}_X) & \rightarrow & \mathrm{Ext}_{\mathcal{A}_Y^e}^2(\tilde{\mathcal{A}}_Y, \tilde{\mathcal{M}}_Y) & \rightarrow & H^2(X, \mathcal{M}_X) & \rightarrow & \\ \dots & & & & & & \end{array}$$

which allows us to analyze the space $\mathrm{exal}(\mathcal{A}_X, \mathcal{M}_X)$. One of the implications is that $\mathrm{exal}(\mathcal{A}_X, \mathcal{M}_X)$ behaves well with respect to base field extensions. It is easy to describe the morphisms

$$H^1(X, \mathcal{M}_X) \rightarrow \mathrm{exal}(\mathcal{A}_X, \mathcal{M}_X) \rightarrow \mathrm{Ext}_{\mathcal{A}_Y^e}^2(\tilde{\mathcal{A}}_Y, \tilde{\mathcal{M}}_Y)$$

explicitly. Note that if X is affine then $H^i(X, \mathcal{M}_X) = 0$ for $i > 0$ and hence $\mathrm{exal}(\mathcal{A}_X, \mathcal{M}_X) = \mathrm{Ext}_{\mathcal{A}_Y^e}^2(\tilde{\mathcal{A}}_Y, \tilde{\mathcal{M}}_Y)$. Moreover, in this case

$$\mathrm{Ext}_{\mathcal{A}_Y^e}^\bullet(\tilde{\mathcal{A}}_Y, \tilde{\mathcal{M}}_Y) = \mathrm{Ext}_{\mathcal{A}_X(X) \otimes \mathcal{A}_X^o(X)}^\bullet(\mathcal{A}_X(X), \mathcal{M}_X(X))$$

and thus

$$\mathrm{exal}(\mathcal{A}_X, \mathcal{M}_X) = \mathrm{exal}(\mathcal{A}_X(X), \mathcal{M}_X(X)).$$

In the special case when $\mathcal{A}_X = \mathcal{O}_X$ and \mathcal{M}_X is a symmetric \mathcal{O}_X -bimodule the isomorphism

$$\mathrm{Ext}_{\mathcal{O}_Y}^i(\mathcal{O}_X, \mathcal{M}_X) = \mathrm{Ext}_{\mathcal{A}_U \otimes \mathcal{A}_U^o}^i(\mathcal{A}_U, \mathcal{M}_U)$$

was proved by R. Swan in [S].

We apply the above results to analyze $\mathrm{def}(\mathcal{A}_X)$ in case X is a smooth quasiprojective variety over \mathbb{C} and $\mathcal{A}_X = D_X$ – the sheaf of differential operators on X . In this case

$$\mathrm{Ext}_{\mathcal{A}_Y^e}^i(\tilde{\mathcal{A}}_Y, \tilde{\mathcal{A}}_Y) = H^i(X^{an}, \mathbb{C}).$$

If in addition X is D -affine (for example X is affine) then $H^i(X, D_X) = 0$ for $i > 0$ and hence

$$\mathrm{def}(D_X) = H^2(X^{an}, \mathbb{C}).$$

In the last section we study *induced* deformations of D_X , i.e. those which come from deformations of the structure sheaf \mathcal{O}_X . In particular if X is a flag variety we show that every induced deformation of D_X is trivial. This result is used in the work [LR3], where we study quantum differential operators on quantum flag varieties.

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2. PRELIMINARIES ON EXTENSION OF ALGEBRAS AND HOCHSCHILD COHOMOLOGY

1. Extensions of algebras. Fix a field k . An algebra means an associative unital k -algebra. Fix an algebra A ; A^o is the opposite algebra and $A^e := A \otimes_k A^o$. An A -module means a left A -module; an A -bimodule means an A^e -module.

Fix an algebra A and an A -bimodule M . Consider an exact sequence of k -modules

$$0 \rightarrow M \rightarrow B \xrightarrow{\epsilon} A \rightarrow 0$$

with the following properties

- B is an algebra and ϵ is a homomorphism of algebras. (Hence M is a 2-sided ideal in B .)
- The B -bimodule structure on M factors through the homomorphism ϵ and the resulting A -bimodule structure on M coincides with the given one. (In particular, the square of the ideal M is zero.)

Definition 2.1. An exact sequence as above is called an *algebra extension* of A by M . An isomorphism between extensions

$$0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$$

and

$$0 \rightarrow M \rightarrow B' \rightarrow A \rightarrow 0$$

is an isomorphism of algebras $\alpha : B \rightarrow B'$ which makes the following diagram commutative

$$\begin{array}{ccccccccc} 0 & \rightarrow & M & \rightarrow & B & \rightarrow & A & \rightarrow & 0 \\ & & id \downarrow & & \alpha \downarrow & & id \downarrow & & \\ 0 & \rightarrow & M & \rightarrow & B' & \rightarrow & A & \rightarrow & 0 \end{array}$$

An extension is *split* if there exists an algebra homomorphism $s : A \rightarrow B$ such that $\epsilon \cdot s = id$. Then $B = A \oplus M$ with the multiplication $(a, m)(a', m') = (aa', am' + ma')$. The collection of isomorphism classes of algebra extensions of A by M is naturally a k -vector space which is denoted $exal(A, M)$. The zero element is the class of the split extension.

Given a map of A -bimodules $M \rightarrow M'$ the usual pushout construction for extensions defines a map

$$exal(A, M) \rightarrow exal(A, M').$$

Given a homomorphism of algebras $A' \rightarrow A$ the pullback construction for extensions defines a map

$$exal(A, M) \rightarrow exal(A', M).$$

Thus $exal(\cdot, \cdot)$ is a bifunctor covariant in the second variable and contravariant in the first one.

In case $M = A$ the space $exal(A, A)$ can be considered as deformations of the first order of the algebra A . Let us describe this space in a different way. Put $k_1 := k[t]/(t^2)$. Consider k_1 -algebras B with a given isomorphism $\theta : grB \rightarrow A \otimes_k k_1$. (The algebra B has the filtration $\{0\} \subset tB \subset B$ and grB denotes the associated graded.) The isomorphism classes of such pairs (B, θ) form a pointed set which we denote by $\text{def}(A)$. The distinguished element in $\text{def}(A)$ is represented by the algebra $B = A \otimes_k k_1$.

We claim that $exal(A, A) = \text{def}(A)$ (hence $\text{def}(A)$ is a k -vector space). Indeed, given (B, θ) as above we obtain an exact sequence

$$0 \rightarrow tB = A \rightarrow B \rightarrow A \rightarrow 0,$$

which gives a well defined map from $\text{def}(A) \rightarrow exal(A, A)$. Conversely, given an algebra extension

$$0 \rightarrow M = A \rightarrow B \rightarrow A \rightarrow 0$$

define the multiplication $t : B \rightarrow B$ by $t \cdot 1_B = 1_A \in M$. This makes B a k_1 -algebra and defines the inverse map $exal(A, A) \rightarrow \text{def}(A)$.

The above description of $exal(A, A)$ allows us to define the set $\text{def}^n(A)$ of n -th order deformations of A as the collection of isomorphism classes of $k_n := k[t]/(t^{n+1})$ -algebras B with an isomorphism of k_n -algebras $grB \rightarrow A \otimes_k k_n$. Thus $\text{def}^1(A) = \text{def}(A) = exal(A, A)$. The algebra $B = A \otimes_k k_n$ represents the *trivial* deformation. Note that B is trivial if there exists a k -algebra homomorphism $s : A \rightarrow B$, which is the left inverse to the residue homomorphism $B \rightarrow A$. Indeed, then $s \otimes 1 : A \otimes_k k_n \rightarrow B$ is an isomorphism of k_n -algebras.

Note that the quotient homomorphism $B \rightarrow B/t^n B$ defines the map $\text{def}^n(A) \rightarrow \text{def}^{n-1}(A)$. Denote by $\text{def}_0^n(A) \subset \text{def}^n(A)$ the preimage in $\text{def}^n(A)$ of the trivial deformation in $\text{def}^{n-1}(A)$.

Lemma 2.2. *There exists a natural identification $\text{def}_0^n(A) = \text{def}(A)$. In particular, $\text{def}_0^n(A)$ has a natural structure of a k -vector space.*

Proof. Let $B \in \text{def}^n(A)$ be such that $B/t^n B = A \otimes_k k_{n-1}$. Consider the obvious k -algebra homomorphism $A \rightarrow A \otimes_k k_{n-1}$ and the induced pullback diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & t^n B & \rightarrow & B' & \rightarrow & A & \rightarrow & 0 \\ & & id \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & t^n B & \rightarrow & B & \rightarrow & A \otimes_k k_{n-1} & \rightarrow & 0 \end{array}$$

Then B' represents an element in $\text{def}(A)$. We get a map $\text{def}_0^n(A) \rightarrow \text{def}(A)$.

The inverse map $\text{def}(A) \rightarrow \text{def}_0^n(A)$ is defined as follows. Given $B' \in \text{def}(A)$ consider the projection $A \otimes_k k_{n-1} \rightarrow A$ and the corresponding pullback diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & A & \rightarrow & B & \rightarrow & A \otimes_k k_{n-1} \rightarrow 0 \\ & & id \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & A & \rightarrow & B' & \xrightarrow{p} & A \rightarrow 0 \end{array}$$

Then B is a k_n -algebra as follows:

$$t : (b', 0) \rightarrow (0, tp(b')), \quad t : (0, t^{n-1}a) \rightarrow (tp^{-1}(a), 0).$$

This proves the lemma. \square

Corollary 2.3. *Assume that $\text{def}(A) = 0$. Then $\text{def}^n(A) = 0$ for all n .*

Proof. Induction on n using the previous lemma. \square

2. Hochschild cohomology. The space $\text{exal}(A, M)$ has a well known cohomological description. Namely, there is a natural isomorphism

$$\text{exal}(A, M) = \text{Ext}_{A^e}^2(A, M).$$

Let us recall how this isomorphism is defined. Consider the bar resolution

$$\dots \xrightarrow{\partial_2} B_1 \xrightarrow{\partial_1} B_0 \xrightarrow{\partial_0} A \rightarrow 0,$$

where $B_i = A^{\otimes i+2}$ and

$$\partial_i(a_0 \otimes \dots \otimes a_{i+1}) = \sum_j (-1)^j a_0 \otimes \dots \otimes a_j a_{j+1} \otimes \dots \otimes a_{i+1}.$$

Note that B_i 's are naturally A^e -modules and the differentials ∂_i are A^e -linear. Hence $B_\bullet \rightarrow A$ is a free resolution of the A^e -module A . Thus for any A^e -module M

$$H^\bullet \text{Hom}_{A^e}(B_\bullet, M) = \text{Ext}_{A^e}^\bullet(A, M).$$

Note that $\text{Hom}_{A^e}(B_i, M) = \text{Hom}_k(A^{\otimes i}, M)$.

Given an algebra extension

$$0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$$

choose a k -linear splitting $s : A \rightarrow B$ and define a 2-cocycle $Z_s \in \text{Hom}_k(A^{\otimes 2}, M)$ by

$$Z_s(a, b) = s(ab) - s(a)s(b).$$

Different k -splittings define cohomologous cocycles, hence we obtain a map $\text{exal}(A, M) \rightarrow \text{Ext}_{A^e}^2(A, M)$ which is, in fact, an isomorphism.

The spaces $\text{Ext}_{A^e}^\bullet(A, M)$ are called the Hochschild cohomology groups of A with coefficients in M . In particular, $\text{Ext}_{A^e}^\bullet(A, A) = HH^\bullet(A)$ is

the usual Hochschild cohomology of A . Note that the space $\text{Ext}_{A^e}^0(A, M) = \text{Hom}_{A^e}(A, M)$ coincides with the center $Z(M)$ of M :

$$Z(M) = \{m \in M \mid am = ma \ \forall a \in A\}.$$

The space $\text{Ext}_{A^e}^1(A, M)$ classifies the outer derivations of A into M . Namely, a map $d : A \rightarrow M$ is a *derivation* if $d(ab) = ad(b) + d(a)b$. It is called an *inner derivation* (defined by $m \in M$) if $d(a) = [a, m]$. Denote by $\text{Der}(A, M)$ (resp. $\text{Inder}(A, M)$) the space of derivations (resp. inner derivations). Then

$$\text{Ext}_{A^e}^1(A, M) = \text{Outder}(A, M) := \text{Der}(A, M) / \text{Inder}(A, M).$$

Remark 2.4. Consider the split extension $B = A \oplus M \in \text{exal}(A)$, i.e. the multiplication in B is $(a, m)(a', m') = (aa', am' + ma')$. Then an automorphism of this extension is an algebra automorphism $\alpha \in \text{Aut}(B)$ of the form

$$\alpha(a, m) = (a, m + d(a)),$$

where $d : A \rightarrow M$ is a derivation. In other words the automorphism group of the trivial extension is the group $\text{Der}(A, M)$.

3. Deformation of sheaves of algebras. Let X be a topological space and \mathcal{A} be a sheaf of k -algebras on X . Let \mathcal{A}^o denote the sheaf of opposite k -algebras and $\mathcal{A}^e = \mathcal{A} \otimes_k \mathcal{A}^o$. Given an \mathcal{A}^e -module \mathcal{M} we may repeat the above definition for algebras and modules to define the space of algebra extensions $\text{exal}(\mathcal{A}, \mathcal{M})$. In particular, an algebra extension of \mathcal{A} by \mathcal{M} is represented by an exact sequence of sheaves of k -vector spaces

$$0 \rightarrow \mathcal{M} \rightarrow \mathcal{B} \xrightarrow{\epsilon} \mathcal{A} \rightarrow 0$$

such that \mathcal{B} is a sheaf of k -algebras and ϵ is a homomorphism of sheaves of algebras satisfying the properties of the Definition 2.1 above. A split extension is the one admitting a homomorphism of sheaves of algebras $s : \mathcal{A} \rightarrow \mathcal{B}$ such that $\epsilon \cdot s = \text{id}$. In particular, a split extension must be split as an extension of sheaves of k -vector spaces.

In case $\mathcal{M} = \mathcal{A}$ we may again define the set $\text{def}^n(\mathcal{A})$ of n -th order deformations of \mathcal{A} , so that $\text{def}^1(\mathcal{A}) = \text{def}(\mathcal{A}) = \text{exal}(\mathcal{A}, \mathcal{A})$. Let again $\text{def}_0^n(\mathcal{A}) \subset \text{def}^n(\mathcal{A})$ be the subset consisting of n -th order deformations which are trivial up to order $n-1$. Then repeating the proof of Lemma 2.2 we get $\text{def}_0^n(\mathcal{A}) = \text{def}(\mathcal{A})$. In particular, $\text{def}_0^n(\mathcal{A})$ is naturally a k -vector space and $\text{def}(\mathcal{A}) = 0$ implies $\text{def}^n(\mathcal{A}) = 0$ for all n .

3. REVIEW OF GERSTENHABER-SCHACK CONSTRUCTION

In the paper [GS] the authors develop a deformation theory of presheaves of algebras on small categories. We will review their construction in a special case which is relevant to us. Namely let X be a topological space and \mathcal{U} be the category of all (or some) open subsets of X . Let $\mathcal{A} = \mathcal{A}_{\mathcal{U}}$ be a presheaf of algebras on \mathcal{U} , i.e. \mathcal{A} is a contravariant functor from \mathcal{U} to the category of algebras. We denote by $k_{\mathcal{U}}$ the constant presheaf of algebras: $k_{\mathcal{U}}(U) = k$ for all $U \in \mathcal{U}$. Let $\mathcal{A} - mod$ be the abelian category (of presheaves) of left \mathcal{A} -modules. The presheaf of algebras $\mathcal{A}^e = \mathcal{A} \otimes \mathcal{A}^o$ is defined in the obvious way: $\mathcal{A}^e(U) = \mathcal{A}(U) \otimes_k \mathcal{A}^0(U)$. In case $\mathcal{A} = k_{\mathcal{U}}$ for $\mathcal{M} \in k_{\mathcal{U}} - mod$ we denote $\text{Ext}_{k_{\mathcal{U}}}^i(k_{\mathcal{U}}, \mathcal{M}) = H^i(\mathcal{U}, \mathcal{M})$.

Fix an \mathcal{A} -bimodule \mathcal{M} (i.e. $\mathcal{M} \in \mathcal{A}^e - mod$). The group $exal(\mathcal{A}, \mathcal{M})$ is defined exactly as above in the case of a single algebra and its bimodule. We are going to give a natural description of the group $exal(\mathcal{A}, \mathcal{M})$ in terms of homological algebra in the category of presheaves on \mathcal{U} . In particular, we will construct a canonical map

$$exal(\mathcal{A}, \mathcal{M}) \rightarrow \text{Ext}_{\mathcal{A}^e}^2(\mathcal{A}, \mathcal{M}).$$

First recall some constructions from [GS].

1. Categorical simplicial resolution. Let $\mathcal{C} = \mathcal{C}_{\mathcal{U}}$ be a presheaf of algebras on \mathcal{U} . Given $U \in \mathcal{U}$ denote its inclusion $i_U : \{U\} \hookrightarrow \mathcal{U}$. The obvious (exact) restriction functor

$$i^* : \mathcal{C} - mod \longrightarrow \mathcal{C}(U) - mod, \quad \mathcal{K} \mapsto \mathcal{K}(U)$$

has a right exact left adjoint functor $i_{U!} : \mathcal{C}(U) - mod \rightarrow \mathcal{C} - mod$

$$i_{U!}K(V) = \begin{cases} \mathcal{C}(V) \otimes_{\mathcal{C}(U)} K, & \text{if } V \subset U, \\ 0, & \text{otherwise.} \end{cases}$$

Thus if K is a projective $\mathcal{C}(U)$ -module, then $i_{U!}K$ is a projective object in $\mathcal{C} - mod$. In particular, the category $\mathcal{C} - mod$ has enough projectives (it also has enough injectives (see [GS])).

If the category \mathcal{U} has a final object U , then $\mathcal{C} = i_{U!}\mathcal{C}(U)$ is projective in $\mathcal{C} - mod$. In particular, then

$$\text{Ext}_{\mathcal{C}}^i(\mathcal{C}, \mathcal{K}) = 0, \quad \text{for all } \mathcal{K} \in \mathcal{C} - mod, i > 0.$$

For $\mathcal{N} \in \mathcal{C} - mod$ define

$$S(\mathcal{N}) := \bigoplus_{U \in \mathcal{U}} i_{U!}i_U^*\mathcal{N}$$

with the canonical map

$$\epsilon_{\mathcal{N}} : S(\mathcal{N}) \rightarrow \mathcal{N}.$$

Clearly S is an endo-functor $S : \mathcal{C} - mod \longrightarrow \mathcal{C} - mod$ with a morphism of functors $\epsilon : S \rightarrow Id$.

Define a diagram of functors

$$\dots s_2 \xrightarrow{\partial_1} s_1 \xrightarrow{\partial_0} s_0 \xrightarrow{\partial_{-1}=\epsilon} Id \rightarrow 0,$$

where $s_i = S^{i+1}$ and $\partial_i = \epsilon_{s_i} - S(\partial_{i-1})$. This diagram is a complex, i.e. $\partial_i \partial_{i-1} = 0$, which is exact. So for $\mathcal{N} \in \mathcal{C} - mod$ we obtain a resolution

$$\dots \rightarrow s_1(\mathcal{N}) \rightarrow s_0(\mathcal{N}) \rightarrow \mathcal{N} \rightarrow 0.$$

Explicitly we have

$$s_k(\mathcal{N}) = \bigoplus_{U_k \subset \dots \subset U_0} i_{U_k!} i_{U_k}^* \dots i_{U_0!} i_{U_0}^* \mathcal{N}.$$

If \mathcal{N} is locally projective (i.e. $\mathcal{N}(U)$ is a projective $\mathcal{C}(U) - module$ for all $U \in \mathcal{U}$), then the complex $s_\bullet(\mathcal{N})$ consists of projective objects in $\mathcal{C} - mod$. So in this case for $\mathcal{M} \in \mathcal{C} - mod$ we have

$$\mathrm{Hom}_{\mathcal{C}}(s_\bullet(\mathcal{N}), \mathcal{M}) = \mathbb{R} \mathrm{Hom}_{\mathcal{C}}^\bullet(\mathcal{N}, \mathcal{M}).$$

2. Simplicial bar resolution. Consider the bar resolution of the presheaf of algebras \mathcal{A} :

$$\dots \rightarrow \mathcal{B}_1 \rightarrow \mathcal{B}_0 \rightarrow \mathcal{A},$$

where $\mathcal{B}_i = \mathcal{A}^{\otimes i+2}$ (this is a direct analogue of the usual bar resolution for algebras described above). The presheaves \mathcal{B}_i are *locally* free \mathcal{A}^e -modules, but usually not projective objects in $\mathcal{A}^e - mod$. So the simplicial resolution $s_\bullet \mathcal{B}_\bullet$ of \mathcal{B}_\bullet is a double complex consisting of projective objects in $\mathcal{A}^e - mod$. For an \mathcal{A}^e -module \mathcal{M} denote by $T^{\bullet\bullet}(\mathcal{M})$ the double complex $\mathrm{Hom}_{\mathcal{A}^e}(s_\bullet \mathcal{B}_\bullet, \mathcal{M})$, and let $T^\bullet(\mathcal{M}) = \mathrm{Tot}(T^{\bullet\bullet}(\mathcal{M}))$ be its total complex. We have

$$\mathrm{Ext}_{\mathcal{A}^e}^i(\mathcal{A}, \mathcal{M}) = H^i(T^\bullet(\mathcal{M})).$$

Consider the double complex $T^{\bullet\bullet}(\mathcal{M})$. It looks like

$$\begin{array}{ccccc} \uparrow & & \uparrow & & \\ \prod_U \mathrm{Hom}_k(\mathcal{A}(U) \otimes \mathcal{A}(U), \mathcal{M}(U)) & \rightarrow & \prod_{V \subset U} \mathrm{Hom}_k(\mathcal{A}(U) \otimes \mathcal{A}(U), \mathcal{M}(V)) & \rightarrow & \\ \uparrow & & \uparrow & & \\ \prod_U \mathrm{Hom}_k(\mathcal{A}(U), \mathcal{M}(U)) & \rightarrow & \prod_{V \subset U} \mathrm{Hom}_k(\mathcal{A}(U), \mathcal{M}(V)) & \rightarrow & \\ \uparrow & & \uparrow & & \\ \prod_U \mathrm{Hom}_k(k, \mathcal{M}(U)) & \rightarrow & \prod_{V \subset U} \mathrm{Hom}_k(k, \mathcal{M}(V)) & \rightarrow & \end{array}$$

where the left lower corner has bidegree $(0, 0)$. The vertical arrows are the Hochschild differentials while the horizontal ones come from the simplicial resolution.

Let $T_a^{\bullet\bullet}(\mathcal{M}) \subset T^{\bullet\bullet}(\mathcal{M})$ be the sub-double complex which is the complement of the bottom row. Put

$$T_a^\bullet(\mathcal{M}) = Tot(T_a^{\bullet\bullet}(\mathcal{M})); \quad H_a^n(\mathcal{A}, \mathcal{M}) := H^n(T_a^\bullet(\mathcal{M})).$$

Note that the complex $T^\bullet(\mathcal{M})/T_a^\bullet(\mathcal{M})$ is just $\text{Hom}_k(s_\bullet(k_{\mathcal{U}}), \mathcal{M})$. Hence we obtain the long exact sequence

$$\begin{aligned} \rightarrow H_a^n(\mathcal{A}, \mathcal{M}) &\rightarrow \text{Ext}_{\mathcal{A}^e}^n(\mathcal{A}, \mathcal{M}) \rightarrow H^n(\mathcal{U}, \mathcal{M}) \\ \rightarrow H_a^{n+1}(\mathcal{A}, \mathcal{M}) &\rightarrow \dots \end{aligned}$$

In case \mathcal{M} is a symmetric \mathcal{A} -bimodule, i.e. $am = ma$ for all $a \in \mathcal{A}$, $m \in \mathcal{M}$, the above sequence splits into short exact sequences ([GS], 21.3)

$$0 \rightarrow H_a^n(\mathcal{A}, \mathcal{M}) \rightarrow \text{Ext}_{\mathcal{A}^e}^n(\mathcal{A}, \mathcal{M}) \rightarrow H^n(\mathcal{U}, \mathcal{M}) \rightarrow 0.$$

3. The isomorphism $exal(\mathcal{A}, \mathcal{M}) \simeq H_a^2(\mathcal{A}, \mathcal{M})$. Let the extension

$$0 \rightarrow \mathcal{M} \rightarrow \mathcal{B} \rightarrow \mathcal{A} \rightarrow 0$$

represent an element in $exal(\mathcal{A}, \mathcal{M})$. Choose local k -linear splittings $s_U : \mathcal{A}(U) \rightarrow \mathcal{B}(U)$. Let us construct a 2-cocycle in $T_a^{\bullet\bullet}(\mathcal{M})$. Namely, put

$$Z^{0,2}(a, b) = s_U(ab) - s_U(a)s_U(b), \quad U \in \mathcal{U}, \quad a, b \in \mathcal{A}(U),$$

$$Z^{1,1}(a) = s_V r_{U,V}^{\mathcal{A}}(a) - r_{U,V}^{\mathcal{B}} s_U(a), \quad V \subset U, \quad a \in \mathcal{A}(U),$$

where $r_{U,V}^{\mathcal{A}} : \mathcal{A}(U) \rightarrow \mathcal{A}(V)$, $r_{U,V}^{\mathcal{B}} : \mathcal{B}(U) \rightarrow \mathcal{B}(V)$ are the structure restriction maps of the presheaves \mathcal{A} and \mathcal{B} . Then $(Z^{0,2}, Z^{1,1})$ is a 2-cocycle in $T_a^{\bullet\bullet}(\mathcal{M})$ and the induced map

$$exal(\mathcal{A}, \mathcal{M}) \rightarrow H_a^2(\mathcal{A}, \mathcal{M})$$

is an isomorphism ([GS], 21.4). The inverse isomorphism is constructed as follows. Let $(Z^{0,2}, Z^{1,1})$ be a 2-cocycle in $T_a^{\bullet\bullet}(\mathcal{M})$. For each $U \in \mathcal{U}$ put $\mathcal{B}(U) = \mathcal{A}(U) \oplus \mathcal{M}(U)$ as a k -vector space; define the multiplication in $\mathcal{B}(U)$ by $(a, m)(a', m') = (aa', am' + ma' + Z^{0,2}(a, a'))$. We make \mathcal{B} the presheaf of algebras by defining the restriction maps $r_{U,V}^{\mathcal{B}} : \mathcal{B}(U) \rightarrow \mathcal{B}(V)$ to be $r_{U,V}^{\mathcal{B}}(a, m) = (r_{U,V}^{\mathcal{A}}(a), r_{U,V}^{\mathcal{M}}(m) + Z^{1,1}(a))$.

In particular, we obtain the 5-term exact sequence

$$\begin{aligned} \dots &\rightarrow \text{Ext}_{\mathcal{A}^e}^1(\mathcal{A}, \mathcal{M}) \rightarrow H^n(\mathcal{U}, \mathcal{M}) \\ \rightarrow exal(\mathcal{A}, \mathcal{M}) &\rightarrow \text{Ext}_{\mathcal{A}^e}^2(\mathcal{A}, \mathcal{M}) \rightarrow H^2(\mathcal{U}, \mathcal{M}) \end{aligned}$$

4. ADMISSIBLE QUASICOHERENT SHEAVES OF ALGEBRAS AND BIMODULES.

Definition 4.1. Let Z be a scheme and \mathcal{A}_Z be a sheaf of unital k -algebras on Z . We say that \mathcal{A}_Z is a *quasicoherent* sheaf of algebras if there is given a homomorphism of sheaves of unital k -algebras $\mathcal{O}_Z \rightarrow \mathcal{A}_Z$ which makes \mathcal{A}_Z a quasicoherent left \mathcal{O}_Z -module. Note that \mathcal{A}_Z° is then a quasicoherent right \mathcal{O}_Z -module. Denote by $\mu(\mathcal{A}_Z) \subset \mathcal{A}_Z\text{-mod}$ the full subcategory of left \mathcal{A}_Z -modules consisting of quasicoherent \mathcal{O}_Z -modules

Fix a quasiprojective scheme X over k with a sheaf of unital k -algebras on \mathcal{A}_X . Let \mathcal{A}_X° be the sheaf of opposite algebras and $\mathcal{A}_X^e = \mathcal{A}_X \otimes_k \mathcal{A}_X^\circ$. An \mathcal{A}_X -module means a left \mathcal{A}_X -module; an \mathcal{A}_X -bimodule means an \mathcal{A}_X^e -module. Put $Y = X \times X$ with the two projections $p_1, p_2 : Y \rightarrow X$. We have the sheaves of algebras $p_1^{-1}\mathcal{A}_X$ and $p_2^{-1}\mathcal{A}_X^\circ$ on Y and hence also their tensor product $p_1^{-1}\mathcal{A}_X \otimes_k p_2^{-1}\mathcal{A}_X^\circ$.

Assume that \mathcal{A}_X is quasicoherent. Then we can take the quasicoherent inverse images $p_1^*\mathcal{A}_X$ and $p_2^*\mathcal{A}_X^\circ$ (using left and right \mathcal{O}_X -structures respectively). Put

$$\mathcal{A}_Y^e := p_1^*\mathcal{A}_X \otimes_{\mathcal{O}_Y} p_2^*\mathcal{A}_X^\circ.$$

Note that for affine open $U, V \subset X$, $\mathcal{A}_Y^e(U \times V) = \mathcal{A}_X(U) \otimes_k \mathcal{A}_X(V)$. This is a quasicoherent sheaf on Y with a natural morphism of quasicoherent sheaves

$$\beta : \mathcal{O}_Y \rightarrow \mathcal{A}_Y^e,$$

which sends 1 to $1 \otimes 1$. We also have the obvious morphism of sheaves of k -vector spaces

$$\gamma : p_1^{-1}\mathcal{A}_X \otimes_k p_2^{-1}\mathcal{A}_X^\circ \rightarrow \mathcal{A}_Y^e.$$

Definition 4.2. We say that the quasicoherent sheaf of algebras \mathcal{A}_X satisfies condition (*) if \mathcal{A}_Y^e has a structure of a sheaf of algebras so that β and γ are morphisms of sheaves of algebras.

Note that if \mathcal{A}_X satisfies condition (*) then, in particular, \mathcal{A}_Y^e is a quasicoherent sheaf of algebras on Y . It seems that the algebra structure on \mathcal{A}_Y^e as required in the condition (*), if it exists, should be unique. In any case, there is a canonical such structure in all examples that we have in mind.

Examples. 1. The condition (*) holds if the sheaf of algebras \mathcal{A}_X is commutative. More generally, if the image of \mathcal{O}_X lies in the center of \mathcal{A}_X .

2. Assume that $\text{char}(k) = 0$ and X is smooth. Then $(*)$ holds for the sheaf $\mathcal{A}_X = D_X$ of differential operators on X . In this case

$$p_1^* D_X \otimes_{\mathcal{O}_Y} p_2^* D_X = D_Y.$$

Let ω_X be the dualizing sheaf on X . Then $D_X^0 = \omega_X \otimes_{\mathcal{O}_X} D_X \otimes_{\mathcal{O}_X} \omega_X^{-1}$ and hence

$$\mathcal{A}_Y^e = p_1^* D_X \otimes_{\mathcal{O}_Y} p_2^* D_X^0 = p_2^* \omega_X \otimes_{\mathcal{O}_Y} D_Y \otimes_{\mathcal{O}_Y} p_2^* \omega_X^{-1}.$$

Let \mathcal{M}_X be an \mathcal{A}_X -bimodule. Then, in particular, \mathcal{M}_X is an \mathcal{O}_X -bimodule.

Definition 4.3. We say that \mathcal{M}_X satisfies the condition (\star) if for an open affine $U \subset X$ and $f \in \mathcal{O}(U)$ we have

$$\mathcal{M}_X(U_f) = \mathcal{O}(U_f) \otimes_{\mathcal{O}(U)} \mathcal{M}_X(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(U_f).$$

Remark 4.4. The sheaves of algebras \mathcal{A}_X in Examples 1,2 above satisfy the condition (\star) when considered as \mathcal{A}_X -bimodules.

Lemma 4.5. *Let \mathcal{A}_X be a quasicoherent sheaf of algebras which satisfies the condition $(*)$, and let \mathcal{M}_X be an \mathcal{A}_X -bimodule which satisfies the condition (\star) . Then \mathcal{M}_X defines a (unique up to an isomorphism) \mathcal{A}_Y^e -module $\tilde{\mathcal{M}}_Y$ on Y such that for an affine open $U \subset X$*

$$\tilde{\mathcal{M}}_Y(U \times U) = \mathcal{M}_X(U).$$

We have $\tilde{\mathcal{M}}_Y \in \mu(\mathcal{A}_Y^e)$.

Proof. Choose an affine open covering $\{U\}$ of X . Then the affine open subsets $U \times U$ form a covering of Y . Fix one such subset $V = U \times U$. The sheaf of algebras \mathcal{A}_Y^e is quasicoherent, hence by Serre's theorem below we have the equivalence of categories

$$\mu(\mathcal{A}_V^e) \simeq \mathcal{A}_V^e(V) - \text{mod}.$$

The sheaf \mathcal{M}_X defines an $\mathcal{A}_Y^e(V) = \mathcal{A}_X(U) \otimes_k \mathcal{A}_X(U)$ -module $\mathcal{M}_X(U)$, hence defines a quasicoherent \mathcal{A}_V^e -module $\tilde{\mathcal{M}}_V$. If $V' = U' \times U' \subset V$, then the condition (\star) for \mathcal{M}_X implies that $\tilde{\mathcal{M}}_V|_{V'} = \tilde{\mathcal{M}}_{V'}$. Hence the local sheaves glue together into a global quasicoherent \mathcal{A}_Y^e -module $\tilde{\mathcal{M}}_Y$. The last assertion is obvious. \square

Theorem 4.6. *Let $Z = \text{Spec} C$ be an affine scheme, \mathcal{A}_Z – a quasicoherent sheaf of algebras on Z . Put $A = \Gamma(X, \mathcal{A}_X)$. Then the functor of global sections Γ is an equivalence of categories*

$$\Gamma : \mu(\mathcal{A}_Z) \rightarrow A - \text{mod}.$$

Its inverse is Δ defined by

$$\Delta(M) = \mathcal{A}_Z \otimes_A M.$$

Both Γ and Δ are exact functors.

Proof. The point is that for an A -module M the quasicoherent sheaf $\Delta(M)$ is indeed an \mathcal{A}_Z -module. The rest follows easily from the classical Serre's theorem about the equivalence

$$qcoh(Z) \simeq C - mod.$$

□

Definition 4.7. We call a quasicoherent sheaf of algebras \mathcal{A}_X *admissible* if it satisfies conditions (*) and (★) (as a bimodule over itself). We call an \mathcal{A}_X -bimodule \mathcal{M}_X *admissible* if it satisfies condition (★). We say that $(\mathcal{A}_X, \mathcal{M}_X)$ is an *admissible pair* if both \mathcal{A}_X and \mathcal{M}_X are admissible.

Remark 4.8. The sheaf of algebras \mathcal{A}_X as in Examples 1,2 above is admissible.

Let us summarize our discussion in the following corollary.

Corollary 4.9. *Let $(\mathcal{A}_X, \mathcal{M}_X)$ be an admissible pair. Then*

i) \mathcal{A}_X defines a quasicoherent sheaf of algebras \mathcal{A}_Y^e on Y such that for affine open $U, V \subset X$, $\mathcal{A}_Y^e(U \times V) = \mathcal{A}_X(U) \otimes_k \mathcal{A}_X(V)^o$;

ii) \mathcal{M}_X defines a sheaf $\tilde{\mathcal{M}}_Y \in \mu(\mathcal{A}_Y^e)$ such that for affine open $U \subset X$, $\tilde{\mathcal{M}}_Y(U \times U) = \mathcal{M}_X(U)$.

Proof. This follows immediately from Definition 4.2 and Lemma 4.5. □

We will be able to give a cohomological interpretation of the group $exal(\mathcal{A}_X, \mathcal{M}_X)$ for an admissible pair $(\mathcal{A}_X, \mathcal{M}_X)$.

5. COHOMOLOGICAL DESCRIPTION OF THE GROUP $exal(\mathcal{A}_X, \mathcal{M}_X)$ FOR AN ADMISSIBLE PAIR $(\mathcal{A}_X, \mathcal{M}_X)$.

Let X be a quasiprojective scheme over k and $(\mathcal{A}_X, \mathcal{M}_X)$ be an admissible pair. We will consider the group $exal(\mathcal{A}_X, \mathcal{M}_X)$ of algebra extensions of \mathcal{A}_X by \mathcal{M}_X . Note that if an exact sequence

$$0 \rightarrow \mathcal{M}_X \rightarrow \mathcal{B}_X \rightarrow \mathcal{A}_X \rightarrow 0$$

is such an extension, then we do not require the sheaf \mathcal{B}_X to be quasicoherent, or even an \mathcal{O}_X -module.

Denote by $\mathcal{U} = \text{Aff}(X)$ be the category of all affine open subsets of X . Given a sheaf \mathcal{F}_X on X we denote by $j_X^* \mathcal{F}_X$ the presheaf on \mathcal{U} , which is obtained by restriction of \mathcal{F}_X to affine open subsets. We will usually denote $j_X^* \mathcal{F}_X = \mathcal{F}_{\mathcal{U}}$ if it causes no confusion. In particular, we obtain presheaves of algebras $\mathcal{A}_{\mathcal{U}} = j_X^* \mathcal{A}_X$, $\mathcal{A}_{\mathcal{U}}^e := \mathcal{A}_{\mathcal{U}} \otimes \mathcal{A}_{\mathcal{U}}^o$ ($\mathcal{A}_{\mathcal{U}}^e \neq j_X^* \mathcal{A}_X^e$).

Lemma 5.1. *Then there is a natural map $\text{exal}(\mathcal{A}_X, \mathcal{M}_X) \rightarrow \text{exal}(\mathcal{A}_{\mathcal{U}}, \mathcal{M}_{\mathcal{U}})$ which is an isomorphism. In particular, $\text{def}(\mathcal{A}_X) = \text{def}(\mathcal{A}_{\mathcal{U}})$.*

Proof. Given an exact sequence of sheaves on X

$$0 \rightarrow \mathcal{M}_X \rightarrow \mathcal{B}_X \rightarrow \mathcal{A}_X \rightarrow 0,$$

which represents an element in $\text{exal}(\mathcal{A}_X, \mathcal{M}_X)$ we obtain the corresponding sequence

$$0 \rightarrow \mathcal{M}_{\mathcal{U}} \rightarrow \mathcal{B}_{\mathcal{U}} \rightarrow \mathcal{A}_{\mathcal{U}} \rightarrow 0$$

of presheaves on \mathcal{U} . This last sequence is exact because \mathcal{M}_X is quasi-coherent. Hence it represents an element in $\text{exal}(\mathcal{A}_{\mathcal{U}}, \mathcal{M}_{\mathcal{U}})$. So we obtain a map

$$\text{exal}(\mathcal{A}_X, \mathcal{M}_X) \rightarrow \text{exal}(\mathcal{A}_{\mathcal{U}}, \mathcal{M}_{\mathcal{U}}).$$

Vice versa, let

$$0 \rightarrow \mathcal{M}_{\mathcal{U}} \rightarrow \mathcal{B}_1 \rightarrow \mathcal{A}_{\mathcal{U}} \rightarrow 0$$

represent an element in $\text{exal}(\mathcal{A}_{\mathcal{U}}, \mathcal{M}_{\mathcal{U}})$. Denote by $^+$ the (exact) functor which associates to a presheaf on \mathcal{U} the corresponding sheaf on X . Then $(\mathcal{A}_{\mathcal{U}})^+ = \mathcal{A}_X$, $(\mathcal{M}_{\mathcal{U}})^+ = \mathcal{M}_X$ and hence we obtain an exact sequence

$$0 \rightarrow \mathcal{M}_X \rightarrow \mathcal{B}_1^+ \rightarrow \mathcal{A}_X \rightarrow 0$$

which defines an element in $\text{exal}(\mathcal{A}_X, \mathcal{M}_X)$. This defines the inverse map

$$\text{exal}(\mathcal{A}_{\mathcal{U}}, \mathcal{M}_{\mathcal{U}}) \rightarrow \text{exal}(\mathcal{A}_X, \mathcal{M}_X).$$

□

Let $D^b(\mathcal{A}_Y^e)$ and $D^b(\mathcal{A}_{\mathcal{U}}^e)$ denote the bounded derived categories of $\mathcal{A}_Y^e - \text{mod}$ and $\mathcal{A}_{\mathcal{U}}^e - \text{mod}$ respectively. Let $D_{\mu(\mathcal{A}_Y^e)}^b(\mathcal{A}_Y^e) \subset D^b(\mathcal{A}_Y^e)$ be the full subcategory consisting of complexes with cohomologies in $\mu(\mathcal{A}_Y^e)$. Denote by $j_Y^* : \mathcal{A}_Y^e - \text{mod} \rightarrow \mathcal{A}_{\mathcal{U}}^e - \text{mod}$ the left exact functor defined by $j_Y^*(\mathcal{F})(U) := \mathcal{F}(U \times U)$, $U \in \mathcal{U}$. Consider its derived functor

$$\mathbb{R}j_Y^* : D^b(\mathcal{A}_Y^e) \rightarrow D^b(\mathcal{A}_{\mathcal{U}}^e).$$

Theorem 5.2. *The functor*

$$\mathbb{R}j_Y^* : D_{\mu(\mathcal{A}_Y^e)}^b(\mathcal{A}_Y^e) \rightarrow D^b(\mathcal{A}_{\mathcal{U}}^e)$$

is fully faithful. Equivalently, for $\mathcal{M}, \mathcal{N} \in \mu(\mathcal{A}_Y^e)$ the map

$$j_Y^* : \text{Ext}_{\mathcal{A}_Y^e}^n(\mathcal{M}, \mathcal{N}) \rightarrow \text{Ext}_{\mathcal{A}_U^e}^n(j_Y^* \mathcal{M}, j_Y^* \mathcal{N})$$

is an isomorphism for all n .

Proposition 5.3. *The map*

$$j_X^* : H^n(X, \mathcal{M}_X) \rightarrow H^n(\mathcal{U}, \mathcal{M}_U)$$

is an isomorphism for all n .

Let us first formulate some immediate corollaries of the theorem and the proposition.

Corollary 5.4. *There exists a natural exact sequence*

$$\text{Ext}_{\mathcal{A}_Y^e}^1(\tilde{\mathcal{A}}_Y, \tilde{\mathcal{M}}_Y) \rightarrow H^1(X, \mathcal{M}_X) \rightarrow \text{exal}(\mathcal{A}_X, \mathcal{M}_X) \rightarrow \text{Ext}_{\mathcal{A}_Y^e}^2(\tilde{\mathcal{A}}_Y, \tilde{\mathcal{M}}_Y) \rightarrow H^2(X, \mathcal{M}_X).$$

In particular, if X is affine then $\text{exal}(\mathcal{A}_X, \mathcal{M}_X) = \text{Ext}_{\mathcal{A}_Y^e}^2(\tilde{\mathcal{A}}_Y, \tilde{\mathcal{M}}_Y)$. If \mathcal{M}_X is a symmetric \mathcal{A}_X -bimodule, then we get a short exact sequence

$$0 \rightarrow \text{exal}(\mathcal{A}_X, \mathcal{M}_X) \rightarrow \text{Ext}_{\mathcal{A}_Y^e}^2(\tilde{\mathcal{A}}_Y, \tilde{\mathcal{M}}_Y) \rightarrow H^2(X, \mathcal{M}_X) \rightarrow 0.$$

Proof. Indeed, this follows from Lemma 5.1, Theorem 5.2, Proposition 5.3 and results of Section 3. \square

Recall the following theorem of J. Bernstein.

Theorem 5.5. ([Bo]) *Let Z be a quasicompact separated scheme, \mathcal{C}_Z – a quasicoherent sheaf of algebras on Z . Then the natural functor*

$$\theta : D^b(\mu(\mathcal{C}_Z)) \rightarrow D_{\mu(\mathcal{C}_Z)}^b(\mathcal{C}_Z)$$

is an equivalence of categories.

Corollary 5.6. *Assume that X is affine. Then*

$$\text{exal}(\mathcal{A}_X, \mathcal{M}_X) \simeq \text{exal}(\mathcal{A}_X(X), \mathcal{M}_X(X)).$$

Proof. Put $\mathcal{A}_X(X) = A$, $\mathcal{M}_X(X) = M$. We have

$$\text{exal}(A, M) = \text{Ext}_{A \otimes A^o}^2(A, M).$$

By Serre's theorem

$$\text{Ext}_{A \otimes A^o}^2(A, M) = \text{Ext}_{\mu(\mathcal{A}_Y^e)}^2(\tilde{\mathcal{A}}_Y, \tilde{\mathcal{M}}_Y).$$

By Bernstein's theorem

$$\text{Ext}_{\mu(\mathcal{A}_Y^e)}^2(\tilde{\mathcal{A}}_Y, \tilde{\mathcal{M}}_Y) = \text{Ext}_{\mathcal{A}_Y^e}^2(\tilde{\mathcal{A}}_Y, \tilde{\mathcal{M}}_Y).$$

Finally, by Corollary 5.4 above

$$\text{Ext}_{\mathcal{A}_Y^e}^2(\tilde{\mathcal{A}}_Y, \tilde{\mathcal{M}}_Y) = \text{exal}(\mathcal{A}_X, \mathcal{M}_X).$$

\square

Question. Under the assumptions of the last corollary let \mathcal{B} be a sheaf of algebras on X representing an element in $\text{exal}(\mathcal{A}_X, \mathcal{M}_X)$. Is $\mathcal{B} = \mathcal{A}_X \oplus \mathcal{M}_X$ as a sheaf of k -vector spaces?

6. PROOF OF THEOREM 5.2 AND PROPOSITION 5.3.

Proof of Proposition 5.3. Let $k_{\mathcal{U}}$ be the constant presheaf on \mathcal{U} and $s_{\bullet}(k_{\mathcal{U}}) \rightarrow k_{\mathcal{U}}$ be its categorical simplicial resolution (Section 3). It is a projective resolution of $k_{\mathcal{U}}$, which consists of direct sums of presheaves $i_{U!}k$. Hence

$$H^i(\mathcal{U}, \mathcal{M}_{\mathcal{U}}) = \text{Ext}^i(k_{\mathcal{U}}, \mathcal{M}_{\mathcal{U}}) = H^i \text{Hom}^{\bullet}(s_{\bullet}(k_{\mathcal{U}}), \mathcal{M}_{\mathcal{U}}).$$

Consider the exact functor $(\cdot)^+$ from the category of presheaves on \mathcal{U} to the category on sheaves on X . Then $k_{\mathcal{U}}^+ = k_X$ – the constant sheaf on X . The functor $(\cdot)^+$ preserves direct sums and $(i_{U!}k)^+ = k_U$ – the extension by zero of the constant sheaf on U . Since \mathcal{M}_X is quasicoherent, for an affine open $U \subset X$ we have $H^i(U, \mathcal{M}_X) = 0$ for all $i > 0$. Thus

$$H^i(X, \mathcal{M}_X) = \text{Ext}^i(k_X, \mathcal{M}_X) = H^i \text{Hom}^{\bullet}(s_{\bullet}(k_{\mathcal{U}})^+, \mathcal{M}_X).$$

It remains to notice that

$$\text{Hom}(k_U, \mathcal{M}_X) = \Gamma(U, \mathcal{M}_X) = \text{Hom}(i_{U!}k, \mathcal{M}_{\mathcal{U}}).$$

This completes the proof of the proposition.

Proof of Theorem 4.2.

Let us formulate a general statement which will imply the theorem. Let Z be a quasicompact separated scheme over k . Let $\text{Aff}(Z)$ be the category of affine open subsets of Z and $\mathcal{W} \subset \text{Aff}(Z)$ be a full subcategory which is closed under intersections and constitutes a covering of Z . Let \mathcal{A}_Z be a quasicoherent sheaf of algebras on Z . Denote by $\mathcal{A}_{\mathcal{W}}$ the corresponding presheaf of algebras on \mathcal{W} . Let

$$j_Z^* : \mathcal{A}_Z - \text{mod} \longrightarrow \mathcal{A}_{\mathcal{W}} - \text{mod}$$

be the natural (left exact) restriction functor.

Proposition 6.1. *In the above notation the derived functor*

$$\mathbb{R}j_Z^* : D_{\mu(\mathcal{A}_Z)}^b(\mathcal{A}_Z) \rightarrow D^b(\mathcal{A}_{\mathcal{W}})$$

is fully faithful.

Proof. By Bernstein's theorem the natural functor

$$\theta : D^b(\mu(\mathcal{A}_Z)) \rightarrow D_{\mu(\mathcal{A}_Z)}^b(\mathcal{A}_Z)$$

is fully faithful. So it suffices to prove that the composition $\mathbb{R}j_Z^* \cdot \theta$ is fully faithful. The functor $j_Z^* : \mu(\mathcal{A}_Z) \rightarrow \mathcal{A}_W - \text{mod}$ is exact. Let $\mathcal{M}, \mathcal{N} \in \mu(\mathcal{A}_Z)$. It suffices to prove that the map

$$j_Z^* : \text{Ext}_{\mu(\mathcal{A}_Z)}^\bullet(\mathcal{M}, \mathcal{N}) \rightarrow \text{Ext}^\bullet(j_Z^*\mathcal{M}, j_Z^*\mathcal{N})$$

is an isomorphism.

Step 1. Assume that Z is affine and $Z \in \mathcal{W}$. Then by Serre's theorem $\mu(\mathcal{A}_Z) \simeq \mathcal{A}_Z(Z) - \text{mod}$. Replacing \mathcal{M} by a left free resolution we may assume that $\mathcal{M} = \mathcal{A}_Z$. But then

$$\text{Ext}^i(\mathcal{A}_Z, \mathcal{N}) = \text{Ext}^i(\mathcal{A}_Z(Z), \mathcal{N}(Z)) = \begin{cases} \mathcal{N}(Z), & \text{if } i = 0 \\ 0, & \text{otherwise} \end{cases}$$

On the other hand $j_Z^*\mathcal{A}_Z = \mathcal{A}_W$ is a projective object in $\mathcal{A}_W - \text{mod}$ (Section 3) and

$$\text{Hom}(\mathcal{A}_W, j_Z^*\mathcal{N}) = \text{Hom}(\mathcal{A}_W(Z), j_Z^*\mathcal{N}(Z)) = \mathcal{N}(Z).$$

So we are done.

Step 2. Reduction to the case when Z is affine.

Let $i_U : U \hookrightarrow Z$ be an embedding of some $U \in \mathcal{W}$. Denote by \mathcal{A}_U the restriction $\mathcal{A}_Z|_U$. We have two (exact) adjoint functors $i_U^* : \mu(\mathcal{A}_Z) \rightarrow \mu(\mathcal{A}_U)$, $i_{U*} : \mu(\mathcal{A}_U) \rightarrow \mu(\mathcal{A}_Z)$. The functor i_{U*} preserves injectives.

Choose a finite covering $Z = \bigcup U_j$, $U_j \in \mathcal{W}$. Then the natural map

$$\mathcal{N} \rightarrow \bigoplus_j i_{U_j*} i_{U_j}^* \mathcal{N}$$

is a monomorphism. So we may assume that $\mathcal{N} = i_{U*} \mathcal{N}_U$ for some $U \in \mathcal{W}$ and $\mathcal{N}_U \in \mu(\mathcal{A}_U)$. Then we have

$$\text{Ext}^\bullet(\mathcal{M}, i_{U*} \mathcal{N}_U) = \text{Ext}^\bullet(i_U^* \mathcal{M}, \mathcal{N}_U).$$

We need a similar construction on the other end. Let $\tilde{i}_U : \mathcal{W}_U \hookrightarrow \mathcal{W}$ be the embedding of the full subcategory $\mathcal{W}_U = \{V \in \mathcal{W} | V \subseteq U\}$. Let $\mathcal{A}_{\mathcal{W}_U}$ be the restriction of \mathcal{A}_W to \mathcal{W}_U . We have the obvious functor $\tilde{i}_U^* : \mathcal{A}_W - \text{mod} \rightarrow \mathcal{A}_{\mathcal{W}_U} - \text{mod}$ and its right adjoint \tilde{i}_{U*} defined by

$$\tilde{i}_{U*}(\mathcal{K})(V) := \mathcal{K}(V \cap U).$$

Both \tilde{i}_U^* and \tilde{i}_{U*} are exact and i_{U*} preserves injectives. For $\mathcal{K} \in \mathcal{A}_{\mathcal{W}_U}$, $\mathcal{L} \in \mathcal{A}_W$ we have

$$\text{Ext}^\bullet(\tilde{i}_U^* \mathcal{L}, \mathcal{K}) = \text{Ext}^\bullet(\mathcal{L}, \tilde{i}_{U*} \mathcal{K}).$$

Note that the following diagrams commute

$$\begin{array}{ccc}
\mu(\mathcal{A}_Z) & \xrightarrow{i_U^*} & \mu(\mathcal{A}_U) \\
j_X^* \downarrow & & \downarrow j_U^* \\
\mathcal{A}_{\mathcal{W}} - \text{mod} & \xrightarrow{\tilde{i}_U^*} & \mathcal{A}_{\mathcal{W}_U} - \text{mod}
\end{array}$$

$$\begin{array}{ccc}
\mu(\mathcal{A}_Z) & \xleftarrow{i_{U*}} & \mu(\mathcal{A}_U) \\
j_X^* \downarrow & & \downarrow j_U^* \\
\mathcal{A}_{\mathcal{W}} - \text{mod} & \xleftarrow{\tilde{i}_{U*}} & \mathcal{A}_{\mathcal{W}_U} - \text{mod}
\end{array}$$

(here j_U^* is the obvious restriction functor). Hence the following diagram commutes as well

$$\begin{array}{ccc}
\text{Ext}^\bullet(\mathcal{M}, \mathcal{N}) & \xrightarrow{j_Z^*} & \text{Ext}^\bullet(j_Z^* \mathcal{M}, j_Z^* \mathcal{N}) = \\
\text{Ext}^\bullet(\mathcal{M}, i_{U*} \mathcal{N}_U) & = & \text{Ext}^\bullet(j_Z^* \mathcal{M}, \tilde{i}_{U*} j_U^* \mathcal{N}_U) = \\
\text{Ext}^\bullet(i_U^* \mathcal{M}, \mathcal{N}_U) & \xrightarrow{j_U^*} & \text{Ext}^\bullet(j_U^* i_U^* \mathcal{M}, j_U^* \mathcal{N}_U).
\end{array}$$

But j_U^* is an isomorphism by Step 1 above. Hence j_Z^* is also an isomorphism. \square

7. A SPECTRAL SEQUENCE

Let X be a quasiprojective variety and $(\mathcal{A}_X, \mathcal{M}_X)$ be an admissible pair. For $\mathcal{N}_1, \mathcal{N}_2 \in \mu(\mathcal{A}_Y^e)$ we will construct a spectral sequence which abuts to $\text{Ext}_{\mathcal{A}_Y^e}^\bullet(\mathcal{N}_1, \mathcal{N}_2)$. In particular we will get an insight into the group $\text{Ext}_{\mathcal{A}_Y^e}^2(\mathcal{A}_Y, \tilde{\mathcal{M}}_Y)$.

Lemma 7.1. *Any object in $\mu(\mathcal{A}_Y^e)$ is a quotient of a locally free \mathcal{A}_Y^e -module.*

Proof. Let $\mathcal{K} \in \mu(\mathcal{A}_Y^e)$. Consider \mathcal{K} as a quasi-coherent \mathcal{O}_Y -module. As such it is a quotient of a locally free \mathcal{O}_Y -module Q (we can take $Q = \oplus \mathcal{O}_Y(-j)$). Then the \mathcal{A}_Y -module $\mathcal{A}_Y \otimes_{\mathcal{O}_Y} Q$ is locally free and surjects onto \mathcal{K} . \square

Let $P_\bullet \rightarrow \mathcal{N}_1$ be a resolution of \mathcal{N}_1 consisting of locally free \mathcal{A}_Y^e -modules. From the proof of the last lemma it follows that there exists an affine covering \mathcal{V} of Y such that for each $V \in \mathcal{V}$ and each P_{-t} the restriction $P_{-t}|_V$ is a free \mathcal{A}_V^e -module. We may (and will) assume that each $V \in \mathcal{V}$ is of the form $U \times U$ for U from an affine open covering \mathcal{U} of X . Choose one such affine covering \mathcal{V} . Let $\check{C}_\bullet(P_\bullet) \rightarrow P_\bullet$ be the corresponding Čech resolution of P_\bullet . This is a double complex

consisting of \mathcal{A}_Y^e -modules, which are extensions by zero from affine open subsets V of free \mathcal{A}_Y^e -modules. Thus

$$H_{\mathcal{A}_Y^e}^\bullet \operatorname{Hom}(\operatorname{Tot}(\check{C}_\bullet(P_\bullet)), \mathcal{N}_2) = \operatorname{Ext}_{\mathcal{A}_Y^e}^\bullet(\mathcal{N}_1, \mathcal{N}_2).$$

The natural filtration of the double complex $\check{C}_\bullet(P_\bullet)$ gives rise to the spectral sequence with the E_2 -term

$$E_2^{p,q} = \check{H}^p(\mathcal{V}, \mathcal{E}xt_{\mathcal{A}_Y^e}^q(\mathcal{N}_1, \mathcal{N}_2)),$$

which abuts to $\operatorname{Ext}_{\mathcal{A}_Y^e}^{p+q}(\mathcal{N}_1, \mathcal{N}_2)$.

In particular, in case $\mathcal{N}_1 = \tilde{\mathcal{A}}_Y^e$, $\mathcal{N}_2 = \tilde{\mathcal{M}}_Y$ this spectral sequence defines a filtration of the group $\operatorname{Ext}_{\mathcal{A}_Y^e}^2(\tilde{\mathcal{A}}_Y^e, \tilde{\mathcal{M}}_Y)$. Namely there are maps

$$\begin{aligned} \alpha_1 : \operatorname{Ext}_{\mathcal{A}_Y^e}^2(\tilde{\mathcal{A}}_Y^e, \tilde{\mathcal{M}}_Y^e) &\rightarrow \check{H}^0(\mathcal{V}, \mathcal{E}xt_{\mathcal{A}_Y^e}^2(\tilde{\mathcal{A}}_Y^e, \tilde{\mathcal{M}}_Y^e)), \\ \alpha_2 : \ker(\alpha_1) &\rightarrow \check{H}^1(\mathcal{V}, \mathcal{E}xt_{\mathcal{A}_Y^e}^1(\tilde{\mathcal{A}}_Y^e, \tilde{\mathcal{M}}_Y^e)), \\ \alpha_3 : \ker(\alpha_2) &\rightarrow \check{H}^2(\mathcal{V}, \mathcal{E}xt_{\mathcal{A}_Y^e}^0(\tilde{\mathcal{A}}_Y^e, \tilde{\mathcal{M}}_Y^e)). \end{aligned}$$

Recall that for $V = U \times U \in \mathcal{V}$ by Bernstein's and Serre's theorems respectively we have

$$\begin{aligned} &\Gamma(V, \mathcal{E}xt_{\mathcal{A}_Y^e}^q(\tilde{\mathcal{A}}_Y^e, \tilde{\mathcal{M}}_Y^e)) \\ &= \Gamma(V, \mathcal{E}xt_{\mu(\mathcal{A}_Y^e)}^q(\tilde{\mathcal{A}}_Y^e, \tilde{\mathcal{M}}_Y^e)) \\ &= \operatorname{Ext}_{\mathcal{A}_X(U) \otimes \mathcal{A}_X^e(U)}^q(\mathcal{A}_X(U), \mathcal{M}_X(U)). \end{aligned}$$

1. Cohomological analysis of the group $\operatorname{exal}(\mathcal{A}_X, \mathcal{M}_X)$. Consider the exact sequence

$$H^1(X, \mathcal{M}_X) \xrightarrow{\epsilon} \operatorname{exal}(\mathcal{A}_X, \mathcal{M}_X) \xrightarrow{\rho} \operatorname{Ext}_{\mathcal{A}_Y^e}^2(\tilde{\mathcal{A}}_Y^e, \tilde{\mathcal{M}}_Y).$$

Let us describe the morphisms ϵ and ρ explicitly.

Since \mathcal{M}_X is quasi-coherent the cohomology group $H^1(X, \mathcal{M}_X)$ is isomorphic to the Čech cohomology $\check{H}^1(\mathcal{U}, \mathcal{M}_X)$. Given a 1-cocycle $\{m_{ij} \in \mathcal{M}_X(U_i \cap U_j) | U_i, U_j \in \mathcal{U}\}$ define an algebra extension

$$0 \rightarrow \mathcal{M}_X \rightarrow \mathcal{B} \rightarrow \mathcal{A}_X \rightarrow 0$$

as follows: on each $U \in \mathcal{U}$ the sheaf $\mathcal{B}|_U$ is a direct sum of sheaves $\mathcal{M}_X|_U$ and $\mathcal{A}_X|_U$ with the multiplication

$$(m, a)(m', a') = (ma' + am', aa').$$

That is, locally \mathcal{B} is a split extension. Define the glueing algebra automorphisms

$$\phi_{ij} : \mathcal{B}_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{B}_{U_i \cap U_j}, \quad \phi_{ij}(m, a) = (m + [a, m_{ij}], a).$$

This defines the map $\epsilon : H^1(X, \mathcal{M}_X) \rightarrow \text{exal}(\mathcal{A}_X, \mathcal{M}_X)$.

Now assume that an algebra extension \mathcal{B} represents an element in $\text{exal}(\mathcal{A}_X, \mathcal{M}_X)$. Consider $\rho(\mathcal{B}) \in \text{Ext}_{\mathcal{A}_Y^e}^2(\tilde{\mathcal{A}}_Y, \tilde{\mathcal{M}}_Y)$ and assume that $\alpha_1(\rho(\mathcal{B})) = 0$, i.e. locally \mathcal{B} is a split extension. Thus for $U \in \mathcal{U}$ we have

$$\mathcal{B}(U) = \mathcal{M}_X(U) \oplus \mathcal{A}_X(U)$$

with the multiplication

$$(m, a)(m', a') = (ma' + am', aa')$$

and with the glueing given by algebra automorphisms

$$\phi_{ij} : \mathcal{B}(U_i \cap U_j) \xrightarrow{\sim} \mathcal{B}(U_i \cap U_j), \quad \phi_{ij}(m, a) = (m + \delta_{ij}(a), a),$$

where $\delta_{ij} : \mathcal{A}_X(U_i \cap U_j) \rightarrow \mathcal{M}_X(U_i \cap U_j)$ is a derivation. For an affine open $U \subset X$ the space

$$\text{Ext}_{\mathcal{A}_X(U) \otimes \mathcal{A}_X^e(U)}^1(\mathcal{A}_X(U), \mathcal{M}_X(U))$$

is the space of outer derivations $\mathcal{A}_X(U) \rightarrow \mathcal{M}_X(U)$. The collection $\{\delta_{ij}\}$ defines an element in $\check{H}^1(\mathcal{V}, \text{Ext}_{\mathcal{A}_Y^e}^1(\tilde{\mathcal{A}}_Y, \tilde{\mathcal{M}}_Y))$, which is equal to $\alpha_2(\rho(\mathcal{B}))$.

Assume now that $\alpha_2(\rho(\mathcal{B})) = 0$. Then there exist elements $\delta_i \in \text{Ext}_{\mathcal{A}_X(U_i) \otimes \mathcal{A}_X(U_i)^0}^1(\mathcal{A}_X(U_i), \mathcal{M}_X(U_i))$ such that $\delta_{ij} = \delta_i - \delta_j$. Changing the local trivializations of \mathcal{B} by the derivations δ_i 's we may assume that δ_{ij} 's are inner derivations. Choose $m_{ij} \in \mathcal{M}_X(U_i \cap U_j)$ so that $\delta_{ij}(a) = [a, m_{ij}]$. The collection $\{m_{ij}\}$ defines a 1-cochain in $\check{C}(\mathcal{U}, \mathcal{M}_X)$. Its coboundary is a 2-cocycle which consists of central elements $m_{ijk} \in \mathcal{M}_X(U_i \cap U_j \cap U_k)$. Thus it defines an element in $\check{H}^2(\mathcal{V}, \text{Hom}_{\mathcal{A}_Y^e}(\tilde{\mathcal{A}}_Y, \tilde{\mathcal{M}}_Y))$. It is equal to $\alpha_3(\rho(\mathcal{B}))$.

8. EXAMPLES

Let X be a smooth complex quasiprojective variety. Let $\delta : X \hookrightarrow Y = X \times X$ be the diagonal embedding, $\Delta = \delta(X)$ – the diagonal, and $p_1, p_2 : Y \rightarrow X$ be the two projections.

1. Deformation of the structure sheaf. Let $\mathcal{A}_X = \mathcal{M}_X = \mathcal{O}_X$. Then $\mathcal{A}_Y^e = \mathcal{O}_Y$, $\tilde{\mathcal{A}}_Y = \delta_* \mathcal{O}_X$. Since the \mathcal{O}_X -bimodule \mathcal{O}_X is symmetric we have the short exact sequence

$$0 \rightarrow \text{def}(\mathcal{O}_X) \rightarrow \text{Ext}_{\mathcal{O}_Y}^2(\delta_* \mathcal{O}_X, \delta_* \mathcal{O}_X) \rightarrow H^2(X, \mathcal{O}_X) \rightarrow 0.$$

Assume that X is projective. By the Hodge decomposition ([GS],[S])

$$\text{Ext}_{\mathcal{O}_Y}^2(\delta_* \mathcal{O}_X, \delta_* \mathcal{O}_X) = H^0(X, \wedge^2 T_X) \oplus H^1(X, T_X) \oplus H^2(X, \mathcal{O}_X).$$

The above short exact sequence identifies $\text{def}(\mathcal{O}_X)$ with $H^0(X, \wedge^2 T_X) \oplus H^1(X, T_X)$. The summand $H^1(X, T_X)$ corresponds to the first order deformations of the variety X by Kodaira-Spencer theory, i.e. to “commutative” deformations of \mathcal{O}_X , while the summand $H^0(X, \wedge^2 T_X)$ corresponds to “noncommutative” deformations.

2. Deformations of the sheaf of differential operators. Let $\mathcal{A}_X = \mathcal{M}_X = D_X$ – the sheaf of (algebraic) differential operators on X . Let ω_X be the dualizing sheaf on X . Then

$$D_X^o = \omega_X \otimes_{\mathcal{O}_X} D_X \otimes_{\mathcal{O}_X} \omega_X^{-1}.$$

We have $D_Y = p_1^* D_X \otimes_{\mathcal{O}_Y} p_2^* D_X$, and hence

$$D_Y^e = p_1^* \omega_X \otimes_{\mathcal{O}_Y} D_Y \otimes_{\mathcal{O}_Y} p_2^* \omega_X^{-1}.$$

The functor $\tau : M \mapsto p_1^* \omega_X \otimes_{\mathcal{O}_Y} M$ is an equivalence of categories

$$\tau : D_Y - \text{mod} \longrightarrow D_Y^e - \text{mod}.$$

Denote by $\delta_+ : D_X - \text{mod} \longrightarrow D_Y - \text{mod}$ the functor of direct image ([Bo]). Then

$$\tilde{D}_Y = \tau(\delta_+ \mathcal{O}_X).$$

Let X^{an} denote the variety X with the classical topology.

Proposition 8.1. *There is a natural isomorphism*

$$\text{Ext}_{D_Y^e}^\bullet(\tilde{D}_Y, \tilde{D}_Y) \simeq H^\bullet(X^{an}, \mathbb{C}).$$

Proof. By the above remarks

$$\text{Ext}_{D_Y^e}^\bullet(\tilde{D}_Y, \tilde{D}_Y) = \text{Ext}_{D_Y}^\bullet(\delta_+ \mathcal{O}_X, \delta_+ \mathcal{O}_X).$$

Let $D_\Delta^b(D_Y)$ be the full subcategory of $D^b(D_Y)$ consisting of complexes with cohomologies supported on Δ . By Kashiwara’s theorem the direct image functor

$$\delta_+ : D^b(D_X) \longrightarrow D_\Delta^b(D_Y)$$

is an equivalence of categories (see [Bo]). Thus, in particular,

$$\text{Ext}_{D_X}^\bullet(\mathcal{O}_X, \mathcal{O}_X) \simeq \text{Ext}_{D_Y}^\bullet(\delta_+ \mathcal{O}_X, \delta_+ \mathcal{O}_X).$$

On the other hand by (a special case of) the Riemann-Hilbert correspondence

$$\text{Ext}_{D_X}^\bullet(\mathcal{O}_X, \mathcal{O}_X) \simeq H^\bullet(X^{an}, \mathbb{C}).$$

□

Corollary 8.2. *Let X be a smooth complex quasi-projective variety. Then we have an exact sequence*

$$H^1(X^{an}, \mathbb{C}) \rightarrow H^1(X, D_X) \rightarrow \text{def}(D_X) \rightarrow H^2(X^{an}, \mathbb{C}) \rightarrow H^2(X, D_X).$$

If X is D -affine (for example X is affine) then

$$\text{def}(D_X) = H^2(X^{an}, \mathbb{C}).$$

Proof. The first part follows immediately from Proposition 8.1 and Corollary 5.4. If X is D -affine, then $H^i(X, D_X) = 0$ for $i > 0$. An affine variety is D -affine since D_X is a quasicoherent sheaf of algebras. This implies the last assertion. \square

Example 8.3. Let $X = \mathbb{C}^n$. Then $\text{def}(D_X) = H^2(X, \mathbb{C}) = 0$. Since X is affine, $\text{def}(D_X) = \text{def}(D_X(X))$, where $D_X(X)$ is the Weyl algebra. It is well known that the Hochschild cohomology of the Weyl algebra is trivial.

9. DEFORMATION OF DIFFERENTIAL OPERATORS

1. Induced deformations of differential operators. Let S be a commutative ring and C be an S -algebra with a finite filtration

$$0 = C_{-1} \subset C_0 \subset C_1 \subset \dots \subset C_n = C,$$

such that the associated graded $gr C$ is commutative. Then it makes sense to define the ring $D_S(C) = D(C)$ of (S -linear) differential operators on C in the usual way. More generally, given two left C -modules M, N define the space of differential operators of order $\leq m$ from M to N as follows.

$$D^m(M, N) = \{d \in \text{Hom}_S(M, N) \mid [f_m, \dots, [f_1, [f_0, d]] \dots] = 0 \text{ for all } f_0, \dots, f_m \in C\}.$$

Then $D(M, N) := \cup_m D^m(M, N)$ and in particular we obtain a filtered (by the order of differential operator) ring $D(C) = D(C, C)$. Note that $C \subset D(C)$ acting by left multiplication. Sometimes we will be more explicit and will write $D({}_C M, {}_C N)$ for $D(M, N)$. If the algebra C is commutative then each k -subspace $D^m(M, N) \subset D(M, N)$ is also a (left and right) C -submodule.

Lemma 9.1. *Denote by S_n the ring $S[t]/(t^{n+1})$. Then canonically*

$$D_{S_n}(C \otimes_S S_n) \simeq D_S(C) \otimes_S S_n.$$

In particular, for a commutative k -algebra A we have

$$D_{k_n}(A \otimes_k k_n) \simeq D_k(A) \otimes_k k_n.$$

Proof. Indeed, every $f \in \text{End}_{S_n}(C \otimes_S S_n) = \text{Hom}_S(C, C \otimes_S S_n)$ can be uniquely decomposed as

$$f = \bigoplus_{i=0}^n f_i \otimes t^i,$$

where $f_i \in \text{End}_S(C, C)$. Now the inclusion $f \in D_{S_n}^m(C \otimes_S S_n)$ is equivalent to inclusions $f_i \in D_S^m(C)$ for all i . Whence the assertion of the lemma. \square

For the rest of this section we will consider only $k[t]$ -algebras, and all differential operators will be $k[t]$ -linear, so we will omit the corresponding subscript. We denote as before $k_n = k[t]/(t^{n+1})$.

Let A be a commutative k -algebra and B be a k_n -algebra with an isomorphism $gr B \simeq A \otimes_k k_n$, i.e. B defines an element in $\text{def}^n(A)$. Consider the inclusion of rings $D(B) \subset \text{End}_{k_n}(B)$. Both these rings are filtered the powers of t , hence we obtain a natural homomorphism (of degree 0 of graded algebras).

$$\alpha : gr D(B) \rightarrow gr \text{End}_{k_n}(B).$$

Note that α may not be injective. On the other hand we have a natural homomorphism of graded algebras

$$\delta : gr \text{End}_{k_n}(B) \rightarrow \text{End}_{k_n}(gr B),$$

which is, in fact, an isomorphism.

We denote the composition of the two maps again by $\gamma : gr D(B) \rightarrow \text{End}_{k_n}(gr B)$.

Lemma 9.2. *i) The homomorphism γ maps $gr D(B)$ to $D(gr B)$.*

ii) The following are equivalent

- a) The map $\gamma : gr D(B) \rightarrow D(gr B)$ is injective*
- b) The map $\gamma : gr D(B) \rightarrow D(gr B)$ is surjective.*

Proof. i). Since everything is k_n -linear, it suffices to prove that $\gamma(D(B)/tD(B)) \subset D(B/tB)$. Let $d \in D^m(B)$ and denote by $\bar{d} \in D(B)/tD(B)$ its residue. Let $b_0, \dots, b_m \in B$ with the corresponding residues $\bar{b}_0, \dots, \bar{b}_m \in B/tB$. We have

$$[b_0, \dots, [b_m, d] \dots] = 0,$$

hence

$$[\bar{b}_0, \dots, [\bar{b}_m, \gamma(\bar{d})] \dots] = 0.$$

Thus $\gamma(\bar{d}) \in D^m(B/tB)$.

ii). The injectivity of $\gamma : gr D(B) \rightarrow D(gr B)$ is equivalent to the injectivity of the natural map $\alpha : gr D(B) \rightarrow gr \text{End}_{k_n}(B)$. Consider the subspace $D(B/tB) \simeq D(B, t^n B) \subset D(B, B)$. The injectivity of

α is equivalent to the assertion that every $d \in D(B, t^n B)$ is equal to $t^n d_1$ for some $d_1 \in D(B)$. But this last assertion is equivalent to the surjectivity of the map $D(B)/tD(B) \rightarrow D(B/tB)$ and hence to the surjectivity if $\gamma : grD(B) \rightarrow D(grB)$. \square

Definition 9.3. Assume that the map $\gamma : grD(B) \rightarrow D(grB)$ is an isomorphism. Then by the Lemma 7.1 the algebra $D(B)$ defines an element in $\text{def}^n(D(A))$. We call $D(B)$ the *induced* (by B) deformation of $D(A)$. We also say that B *induces* a deformation of $D(A)$.

Example 9.4. It follows from Lemma 7.1 that the trivial deformation of A induces a deformation of $D(A)$, which is also trivial.

Remark 9.5. It would be interesting to see which deformations of A induce deformations of $D(A)$.

2. Two lemmas about induced deformations. Assume that A and B are as above and B induces a deformation of $D(A)$. Denote the residue map $\tau : D(B) \rightarrow D(A)$. Moreover, assume that $D(B)$ is a split extension of $D(A)$ with a splitting homomorphism (of k -algebras) $s : D(A) \rightarrow D(B)$. Since $A \subset D(A)$, the map s defines, in particular, a structure of a left $A \otimes_k k_n$ -module on B . The next two lemmas will be used in what follows.

Lemma 9.6. *i) The residue map $\beta : B \rightarrow A$ is a homomorphism of left A -modules.*

ii) B is a free $A \otimes_k k_n$ -module of rank 1.

Proof. i). Given $a \in A$, $b \in B$ we need to show that $\beta(s(a)b) = a\beta(b)$. This follows from the identity $\tau s(a) = a$ and the commutativity of the diagram

$$\begin{array}{ccc} D(B) \times B & \xrightarrow{(\tau, \beta)} & D(A) \times A \\ \downarrow & & \downarrow \\ B & \xrightarrow{\beta} & A, \end{array}$$

where the vertical arrows are the action morphisms.

ii) The A -module map $\beta : B \rightarrow A$ has a splitting $\alpha : A \rightarrow B$, which induces an isomorphism $\alpha \otimes 1 : A \otimes_k k_n \rightarrow B$ of left $A \otimes_k k_n$ -modules. \square

Lemma 9.7. *Assume that the k -algebra A is finitely generated. Consider B with the structure of a left $A \otimes_k k_n$ -module defined above. Then $D(B) = D(A \otimes_k k_n B)$ as subrings of $\text{End}_{k_n}(B)$.*

Proof. Denote $\tilde{A} = A \otimes_k k_n$. Since $D(B)$ is a deformation of $D(A)$ the graded ring $grD(B)$ coincides with the subring $D(grB) \subset \text{End}_{k_n}(grB)$. The isomorphism of \tilde{A} -modules ${}_{\tilde{A}}B \simeq \tilde{A}$ defines an isomorphism of rings

$$D({}_{\tilde{A}}B) \simeq D(\tilde{A}) = D(grB).$$

Hence, in particular, $grD({}_{\tilde{A}}B)$ is a graded submodule of $\text{End}_{k_n}(grB)$ and as such coincides with $D(grB)$. We conclude that the graded subrings of $\text{End}_{k_n}(grB)$, $grD(B)$ and $grD({}_{\tilde{A}}B)$ coincide ($= D(grB)$). So it suffices to prove the inclusion $D({}_B B) \subset D({}_{\tilde{A}}B)$.

We will prove by descending induction on p that

$$D({}_B B, {}_B t^p B) \subset D({}_{\tilde{A}} B, {}_{\tilde{A}} t^p B).$$

It follows from Lemma 7.6,i) that the A - and B -module structure on B coincide modulo t . More precisely, if $b \in B$ and $a = \beta(b) \in A$, then

$$s(a) - b : t^\bullet B \rightarrow t^{\bullet+1} B.$$

This implies that

$$D({}_B B, {}_B t^n B) = D({}_{\tilde{A}} B, {}_{\tilde{A}} t^n B).$$

Suppose that we proved the inclusion $D({}_B B, {}_B t^{p+1} B) \subset D({}_{\tilde{A}} B, {}_{\tilde{A}} t^{p+1} B)$. Let a_1, \dots, a_l be a set of generators of the algebra A . Choose $d \in D^m({}_B B, {}_B t^p B)$. Then the operators

$$d_{i_0 \dots i_m} := [s(a_{i_0}), \dots, [s(a_{i_m}), d] \dots], \quad i_j \in \{1, \dots, l\}$$

map B to $t^{p+1} B$. Since $s(a_i) \in D({}_B B)$ also

$$d_{i_0 \dots i_m} \in D({}_B B, {}_B t^{p+1} B) \subset D({}_{\tilde{A}} B, {}_{\tilde{A}} t^{p+1} B).$$

Thus there exists N such that every $d_{i_0 \dots i_m} \in D^N({}_{\tilde{A}} B, {}_{\tilde{A}} t^{p+1} B)$. Since A is commutative this implies that for any $c_1, \dots, c_m \in A$

$$[s(c_1), \dots, [s(c_m), d] \dots] \in D^N({}_{\tilde{A}} B, {}_{\tilde{A}} t^{p+1} B).$$

But then $d \in D^{N+m}({}_{\tilde{A}} B, {}_{\tilde{A}} t^p B)$. Hence $D({}_B B, {}_B t^p B) \subset D({}_{\tilde{A}} B, {}_{\tilde{A}} t^p B)$, which completes the induction step and proves the lemma. \square

3. Sheafification. Let Y be a scheme over k , \mathcal{B} – a sheaf of k_n -algebras on Y with an isomorphism of sheaves of k_n -algebras $gr\mathcal{B} \simeq \mathcal{O}_Y \otimes_k k_n$, i.e. \mathcal{B} defines an element in $\text{def}^n(\mathcal{O}_Y)$. Then using the commutator definition as in 9.1 above we define the sheaf $D(\mathcal{B})$ of k_n -linear differential operators on \mathcal{B} . Thus, in particular, $D(\mathcal{B})$ is a subsheaf of $\mathcal{E}nd_{k_n}(\mathcal{B})$. In this section all the differential operators will be $k[t]$ -linear, so we omit the corresponding subscript.

As in the ring case we obtain a natural homomorphism of sheaves of graded k_n -algebras (which, probably, is neither injective, nor surjective in general)

$$\tilde{\gamma} : grD(\mathcal{B}) \rightarrow \mathcal{E}nd_{k_n}(gr\mathcal{B}).$$

The following two lemmas are the sheaf versions of Lemmas 9.1 and 9.2 which will be used later. The proofs are the same.

Lemma 9.8. $D(\mathcal{O}_Y \otimes_k k_n) = D(\mathcal{O}_Y) \otimes_k k_n$ ($= D_Y \otimes_k k_n$).

Lemma 9.9. *The homomorphism $\tilde{\gamma}$ maps $grD(\mathcal{B})$ to $D(gr\mathcal{B})$.*

Definition 9.10. Assume that $\tilde{\gamma} : grD(\mathcal{B}) \rightarrow D(gr\mathcal{B})$ is an isomorphism. Then by Lemma 9.8 the sheaf $D(\mathcal{B})$ defines an element in $\text{def}^n(D_Y)$. We call $D(\mathcal{B})$ the *induced* (by \mathcal{B}) deformation of D_Y and say that \mathcal{B} *induces* this deformation.

4. Deformations of differential operators on a flag variety.

Theorem 9.11. *Let G be a complex linear simple simply connected algebraic group, $B \subset G$ – a Borel subgroup, $X = G/B$ – the corresponding flag variety. Then any induced deformation of D_X is trivial.*

Remark 9.12. Since $H^1(X, T_X) = 0$ (the variety X is rigid) the only deformations of \mathcal{O}_X are “purely noncommutative”, i.e. they correspond to elements of $H^0(X, \wedge^2 T_X)$. In this respect one may ask the following question: Suppose Y is a smooth projective variety, \mathcal{B} – a purely noncommutative deformation of \mathcal{O}_Y . Assume that \mathcal{B} induces a deformation $D(\mathcal{B})$ of D_Y . Is $D(\mathcal{B})$ a trivial deformation of D_Y ?

Proof. Assume that a sheaf of k_n -algebras \mathcal{B} , which represents an element in $\text{def}^n(\mathcal{O}_X)$, induces a deformation (of order n) $D(\mathcal{B})$ of D_X . Then for any $m > 0$ the sheaf $\mathcal{B}/t^{m+1}\mathcal{B}$ induces a deformation of order m , $D(\mathcal{B}/t^{m+1}\mathcal{B})$, of D_X . By induction we may assume that $D(\mathcal{B}/t^n\mathcal{B}) \simeq D_X \otimes_k k_{n-1}$, i.e. $D(\mathcal{B})$ represents an element in $\text{def}_0^n(D_X)$. (Recall that $\text{def}_0^n(D_X) \simeq \text{def}(D_X)$.) We need to prove that $D(\mathcal{B})$ is the trivial element in $\text{def}^n(D_X)$. For simplicity of notation we assume that $n = 1$ (the proof in the general case is the same).

It is well known that X has an open covering $X = \cup_{w \in W} U_w$, where W is the Weyl group of G and $U_w \simeq \mathbb{C}^d$, $d = \dim(X)$. Denote the covering $\mathcal{U} = \{U_w\}$. It follows from Example 8.3 that $D(\mathcal{B})_{U_w}$ is the trivial deformation of D_{U_w} for each $w \in W$.

The variety X is D -affine ([BB]), thus $\text{def}(D_X) \simeq H^2(X^{an}, \mathbb{C})$ (Corollary 8.2). But $H^2(X^{an}, \mathbb{C}) = H^{1,1}(X^{an}, \mathbb{C}) = \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}$. Let us describe the isomorphism $\sigma : \text{Pic}(X) \otimes \mathbb{C} \rightarrow \text{def}(D_X)$ directly. Let \mathcal{L}

be a line bundle on X . Then $\mathcal{L}|_{U_w} \simeq \mathcal{O}_{U_w}$ for all $w \in W$. Hence \mathcal{L} is defined by a Čech 1-cocycle $\{f_{ij} \in \mathcal{O}_{U_{w_i} \cap U_{w_j}}^*\}$. Define derivations

$$\delta_{ij} : D_{U_{w_i} \cap U_{w_j}} \rightarrow D_{U_{w_i} \cap U_{w_j}}$$

by the formula

$$\delta_{ij}(d) = [d, \log(f_{ij})].$$

Note that though $\log(f_{ij})$ is a multivalued analytic function, $[\cdot, \log(f_{ij})]$ is a well defined derivation of the ring of differential operators and it preserves the algebraic operators. So δ_{ij} is well defined. Using these derivations we define the glueing over $U_{w_1} \cap U_{w_2}$ of the sheaves $D_{U_{w_i}} \otimes \mathbb{C}[t]/(t^2)$ and $D_{U_{w_j}} \otimes \mathbb{C}[t]/(t^2)$. We denote the corresponding global sheaf $\sigma(\mathcal{L})$. The map $\sigma : \text{Pic}(X) \rightarrow \text{def}(D_X)$ is a group homomorphism which extends to an isomorphism

$$\sigma : \text{Pic}(X) \otimes \mathbb{C} \xrightarrow{\sim} \text{def}(D_X).$$

Let us get back to $D(\mathcal{B}) \in \text{def}(D_X)$. By the above isomorphism, $D(\mathcal{B}) = \sigma(\mathcal{L})$ for some $\mathcal{L} \in \text{Pic}(X) \otimes \mathbb{C}$. We have $D(\mathcal{B})_{U_w} = D_{U_w} \otimes \mathbb{C}[t]/(t^2)$, so that \mathcal{B}_{U_w} has a structure of a D_{U_w} -module and, in particular, of an \mathcal{O}_{U_w} -module. By (a sheaf version of) Lemma 9.6,ii) $\mathcal{B}_{U_w} \simeq \mathcal{O}_{U_w} \otimes \mathbb{C}[t]/(t^2)$ as an \mathcal{O}_{U_w} -module. Since the glueing of different $D(\mathcal{B})_{U_w}$'s is by means of derivations $[\cdot, \log(f_{ij})]$, it follows that the local \mathcal{O}_{U_w} -module structure on \mathcal{B} agree on the intersections $U_{w_i} \cap U_{w_j}$. Hence \mathcal{B} is an \mathcal{O}_X -module, which fits in the short exact sequence of \mathcal{O}_X -modules

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{B} \rightarrow \mathcal{O}_X \rightarrow 0.$$

Since $\text{Ext}_{\mathcal{O}_X}^1(\mathcal{O}_X, \mathcal{O}_X) = 0$, $\mathcal{B} = \mathcal{O}_X \otimes \mathbb{C}[t]/(t^2)$. Thus $D(\mathcal{O}_X \mathcal{B}) = D_X \otimes \mathbb{C}[t]/(t^2)$. But by (a sheaf version of) Lemma 9.7 $D(\mathcal{O}_X \mathcal{B}) = D(\mathcal{B}) (= D(\mathcal{B}))$, which proves the theorem. \square

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